

# Curvature representation of the gonihedric action

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## Abstract

We analyse the curvature representation of the gonihedric action  $A(M)$  for the cases when the dependence on the dihedral angle is arbitrary.

# 1 Introduction

There is a strong belief that QCD is equivalent to a string theory and that this equivalence could play an important role in the development of the theory of strong interactions [1, 2, 3, 4, 5]. The solution of the two-dimensional QCD in terms of strings has been invoked to put forward the ultimate conjecture that QCD is actually equivalent to string theory. The  $QCD_2$  string has the property to suppress folding of the surface and therefore has a kind of extrinsic curvature [1].

In the recent articles [6, 7, 8] the authors suggested a geometrical string, which directly extends the notion of the Feynman integral over paths to an integral over surfaces. In this theory the surface which degenerates into a single particle world line has an amplitude which is proportional to the length of world line. This approach was motivated by the fact that in the first approximation the motion of the particle with internal structure is described by the action which should be proportional to the length of the spacetime path [8]. This principle, together with the continuity principle allows to define the action of the theory  $A(M)$ . For the smooth surfaces this action is proportional to the mean extrinsic curvature and has the property to suppress the folding of the surface [8]. The degree of suppression depends on the gonimetric factor  $\Theta$  in the definition of the action  $A(M)$  and can be made arbitrary large. For that the  $\Theta$  should grow fast enough near the dihedral angles close to  $\pi$  which describe a flat surfaces [8].

In the subsequent publication [9] the authors have found that the action  $A(M)$  has an equivalent representation in terms of the total curvature  $k(E)$  of the polygons  $C_E$  which appear in the intersection of the plane  $E$  with the given surface  $M$ . The curvature  $k(E)$  should then be integrated over all planes  $E$  intersecting the surface  $M$ . This representation has been proven for the special case of the action  $A(M)$ , when the action depends linearly on dihedral angle  $\alpha$ . To provide necessary rigidity of the polygons  $\{C_E\}$  and the corresponding suppression of the folding of the surface it is important to have generalization of the curvature representation for the cases when the dependence on the dihedral angle  $\alpha$  is arbitrary [8]. The aim of this article is to perform numerical analyses of the curvature representation of the gonihedric action for the cases when dependence on the dihedral angle is arbitrary.

# 2 The Action

The gonihedric string has been formulated in an Euclidean space  $R^d$ , where the Feynman integral coincides with the partition function for the randomly fluctuating surfaces. In the continuum Euclidean space  $R^d$  the random surface is associated with a connected polyhedral surface  $M$  with vertex coordinates  $X_i$ , where  $i = 1, \dots, |M|$  and  $|M|$  is the number of the vertices. The action is defined as [6]

$$A(M) = \sum_{\langle ij \rangle} |X_i - X_j| \cdot \Theta(\alpha_{ij}), \quad (1)$$

where the gonimetric factor  $\Theta$  is defined as

$$\Theta(\pi) = 0, \quad \Theta(\alpha) \geq 0, \quad (2)$$

$$\Theta(2\pi - \alpha) = \Theta(\alpha), \quad (3)$$

the summation is over all edges  $\langle ij \rangle$  of  $M$  and  $\alpha_{ij}$  is a dihedral angle between two neighbor faces (flat triangles) of  $M$  in  $R^d$  having a common edge  $\langle ij \rangle$ . The property (3) always allows to consider the argument of the  $\Theta$  function to be in the interval  $(0, \pi)$  in all subsequent computations. One can say that we are measuring the dihedral angle  $\alpha_{ij}$  from that side of the surface where it is less than  $\pi$ . As it is easy to see the action indeed has the physical dimension of the length and the degree of the suppression of the folding of the surface depends on  $\Theta$  [8]. In cases when the  $\Theta$  is equal to

$$\Theta = |\pi - \alpha|^\varsigma$$

and  $\varsigma < 1$  the suppression can be made arbitrary large. The physical properties of this model and possible connection with QCD have been discussed in [6, 7, 8].

As we stress above the new representation of action  $A(M)$  have been proven [9] for the special case when the factor  $\Theta$  is linear

$$\Theta(\alpha) = |\pi - \alpha|. \quad (4)$$

In the present article we shall perform the numerical analyses of the curvature representation for the cases when the dependence on the dihedral angle is arbitrary. The numerical integration shows that one can always select the dependence from the dihedral angle so as to provide necessary rigidity of the polygons  $\{C_E\}$  and therefore corresponding suppression of the folding of the surfaces. This may help to proceed with the analytical results for the general case.

### 3 Curvature representation

If we denote by  $\alpha_{ij}^E$  the angle which appears in the intersection of the plane  $E$  with dihedral angle  $\alpha_{ij}$  of the edge  $\langle ij \rangle$ , then the total curvature  $k(E)$  of the polygon or polygons appearing in the intersection is equal to the following sum [9]

$$k(E) = \sum_{\langle ij \rangle} |\pi - \alpha_{ij}^E|. \quad (5)$$

Here we imply that  $\alpha_{ij}^E$  is equal to  $\pi$  for the edges  $\langle ij \rangle$  which are not intersected by the given plane  $E$ . With this definitions the gonihedric action  $A(M)$  is equal to [9]

$$A(M) = \int k(E)dE, \quad (6)$$

where  $dE$  is the invariant measure of two-dimensional planes  $E$  in  $R^3$  [12, 13, 14]. This measure can be defined explicitly in  $R^3$  through the coordinates of the unit vector  $\vec{n}(\theta, \phi)$  which is normal to the plane  $E$  and through the distance  $\rho$  of the plane  $E$  from the origin [12]

$$dE = \frac{1}{2\pi} d\rho \sin \theta d\theta d\phi. \quad (7)$$

To prove the representation (6) let us substitute (5) and (7) into (6), this yields [9]

$$\begin{aligned} A(M) &= \int \sum_{\langle ij \rangle} |\pi - \alpha_{ij}^E| dE = \sum_{\langle ij \rangle} \int |\pi - \alpha_{ij}^E| dE \\ &= \sum_{\langle ij \rangle} \frac{1}{2\pi} \int |\pi - \alpha_{ij}^E| \int_0^{|X_i - X_j||\cos \theta|} d\rho \sin \theta d\theta d\phi \\ &= \sum_{\langle ij \rangle} |X_i - X_j| \cdot \langle |\pi - \alpha_{ij}^E| \rangle, \end{aligned} \quad (8)$$

where

$$\langle |\pi - \alpha^E| \rangle = \frac{1}{2\pi} \int |\pi - \alpha^E| |\cos \theta| \sin \theta d\theta d\phi \equiv \Theta(\alpha). \quad (9)$$

Now one should prove, that  $\Theta(\alpha)$  in (9), after averaging over all orientations of the plane  $E$  is indeed equal to  $|\pi - \alpha|$  (4). This has been proven in [9], but here we will present a new proof which allows to apply numerical integration to more complicated cases.

We have the following expression for  $|\pi - \alpha^E|$

$$\begin{aligned} |\pi - \alpha^E| &= \arccos \left( -\frac{|\vec{r}_1 \vec{r}_2|}{|\vec{r}_1| \cdot |\vec{r}_2|} \right) \\ &= \arccos \left( -\frac{\cos(\alpha) + \tan^2(\theta) \cos(\phi) \cos(\phi - \alpha)}{\sqrt{1 + \tan^2(\theta) \cos^2(\phi)} \cdot \sqrt{1 + \tan^2(\theta) \cos^2(\phi - \alpha)}} \right), \end{aligned} \quad (10)$$

where  $\alpha^E = \alpha^E(\alpha, \theta, \phi)$  is the angle in the intersection of the plane  $E$  with dihedral angle  $\alpha$ .

It is very hard to integrate directly (9) and (10) to get (4). Instead of that we would like to remark, that from (9) and (10) it follows that

$$\Theta(\pi) = 0, \quad \Theta(0) = \pi \quad (11)$$

and that

$$\Theta(\pi - \alpha) = \pi - \Theta(\alpha) \equiv \langle \alpha^E \rangle, \quad (12)$$

where as before we imply that all angles are in the interval  $(0, \pi)$  (see property (2)).

The basic property of the dihedral angle  $\alpha^E$  is the *additivity* property

$$\alpha^E(\alpha_1 + \alpha_2, \theta, \phi) = \alpha^E(\alpha_1, \theta, \phi) + \alpha^E(\alpha_2, \theta, \phi - \alpha_1) \quad (13)$$

which follows from the geometrical picture or one can check it directly using explicit formula (10).

Now substituting (13) into (9) and using (12) we will get

$$\Theta(\alpha_1 + \alpha_2) = \Theta(\alpha_1) + \Theta(\alpha_2) - \pi. \quad (14)$$

The last functional equation together with the boundary condition (11) has unique solution

$$\Theta(\alpha) = < |\pi - \alpha^E| > = |\pi - \alpha|. \quad (15)$$

This completes the proof of the curvature representation (6) and (5) for  $A(M)$  in the particular case (4).

## 4 Extended curvature

Our aim is to extend this result into the cases when  $\Theta(\alpha)$  is an arbitrary function of  $|\pi - \alpha|$ . Generally one can expect that

$$A(M) = \int k_\tau(E) dE = \sum_{<ij>} |X_i - X_j| \cdot \Theta(\alpha_{ij}), \quad (16)$$

where the extended curvature  $k_\tau(E)$  is defined as

$$k_\tau(E) = \sum_{<ij>} \tau(|\pi - \alpha_{ij}^E|) \quad (17)$$

and the function  $\Theta$  is defined by the integration over all planes  $E$

$$\Theta(\alpha) = \frac{1}{2\pi} \int \tau(|\pi - \alpha^E|) |\cos(\theta)| \sin(\theta) d\theta d\phi. \quad (18)$$

Now the problem is to select an appropriate function  $\tau$  in the definition of the total curvature  $k_\tau(E)$  such that the integration over all planes  $E$  will produce the goniometric factor  $\Theta$  with the necessary properties (2) and (3).

The most important cases are when  $\tau$  is a power of  $|\pi - \alpha^E|$

$$\tau = |\pi - \alpha^E|^n, \quad n = 1, 2, \dots \quad (19)$$

or a root function

$$\tau = |\pi - \alpha^E|^{1/n}, \quad n = 1, 2, \dots \quad (20)$$

It is also interesting to compute the two angle correlation function

$$\Theta_{nm}(\alpha_1, \alpha_2) = < |\pi - \alpha^E(\alpha_1)|^n \cdot |\pi - \alpha^E(\alpha_2)|^m > \quad (21)$$

which is defined by the integral

$$\Theta_{nm}(\alpha_1, \alpha_2) = \frac{1}{2\pi} \int |\pi - \alpha^E(\alpha_1, \theta, \phi)|^n |\pi - \alpha^E(\alpha_2, \theta, \phi - \alpha_1)|^m |\cos \theta| \sin \theta d\theta d\phi \quad (22)$$

As it is easy to see from (9),(10) and from the last integral it is almost impossible to compute these explicitly, therefore we should use the numerical integration to study their behaviour.

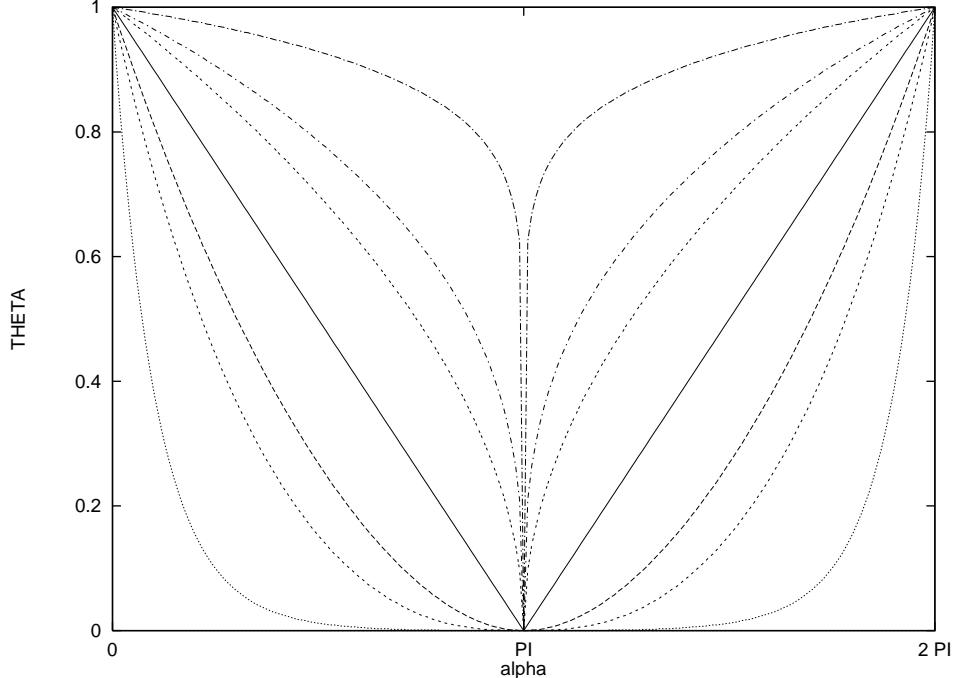


Figure 1: Gonimetric function  $\Theta$  for the cases when  $\tau$  is given by (19) and (20) and  $n=1,2,3,10$ . The lowest curve corresponds to  $n = 10$  in (19) and the uppermost curve to  $n = 10$  in (20). The curves are normalized to unity at  $\alpha = 0$ .

## 5 Numerical results

In figure 1 we show the result of the integration over all planes  $\{E\}$  in (18) for various functions  $\tau$ . As we see the qualitative behaviour of the gonimetric function  $\Theta$  is very similar to the behaviour of the corresponding  $\tau$  function.

Therefore it is reasonable to ask how big is the “deformation” of the functions (19) and (20) after integration over all planes  $\{E\}$  in (18), that is the magnitude of the differences

$$\langle |\pi - \alpha^E|^n \rangle - |\pi - \alpha|^n, \quad (23)$$

or

$$\langle |\pi - \alpha^E|^{1/n} \rangle - |\pi - \alpha|^{1/n}, \quad (24)$$

In figure 2 one can see that the differences are not very large, but the equality between these two quantities is definitely excluded. Only for  $n = 1$  does the equality hold [9]. The question to which we cannot get an answer by numerical analyses is how to select the function  $\tau$  in (17) to recover the power or root behaviour of the  $\Theta$  function, that is to find out the coefficients  $c_k$  in the  $\tau$  series

$$\tau = \sum_k c_k |\pi - \alpha^E|^k \quad (25)$$

such that

$$\langle \sum_k c_k |\pi - \alpha^E|^k \rangle = |\pi - \alpha|^n. \quad (26)$$

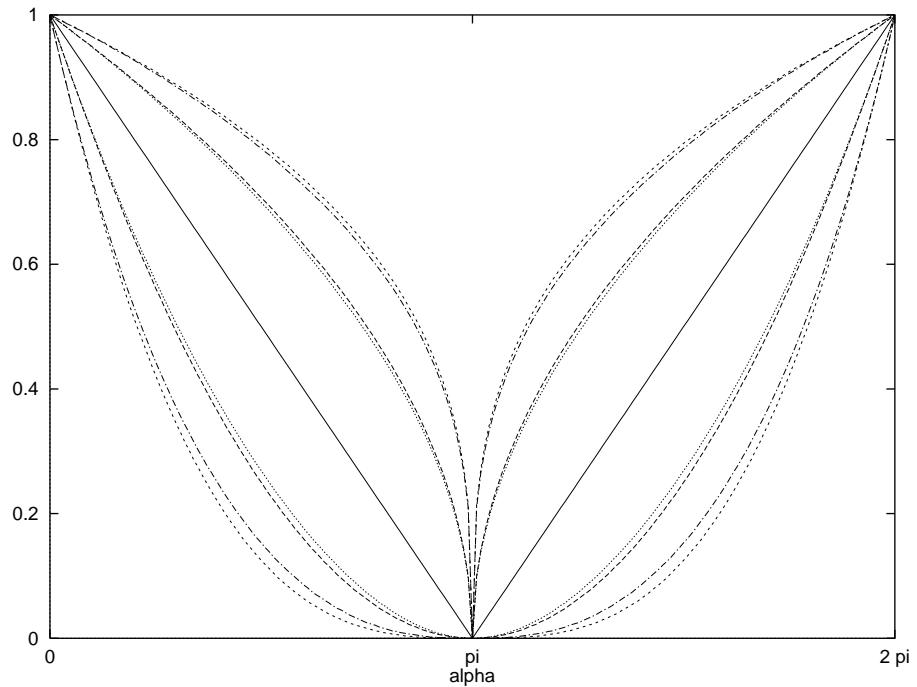
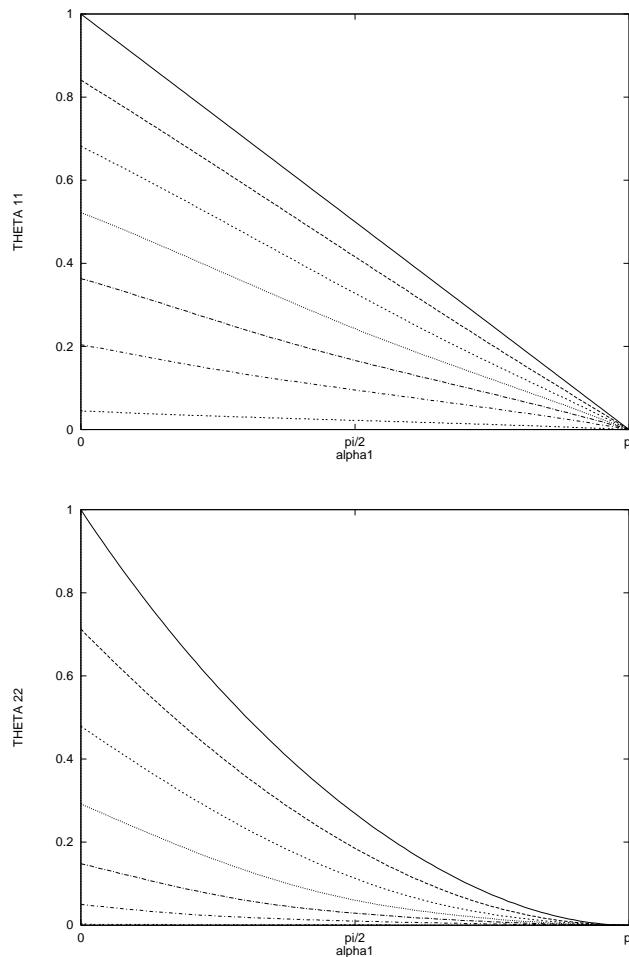


Figure 2: Comparison of the gonimetric function  $\Theta$  against  $|\pi - \alpha|^n$  for  $n=1, 2, 3, 1/2, 1/3$ .



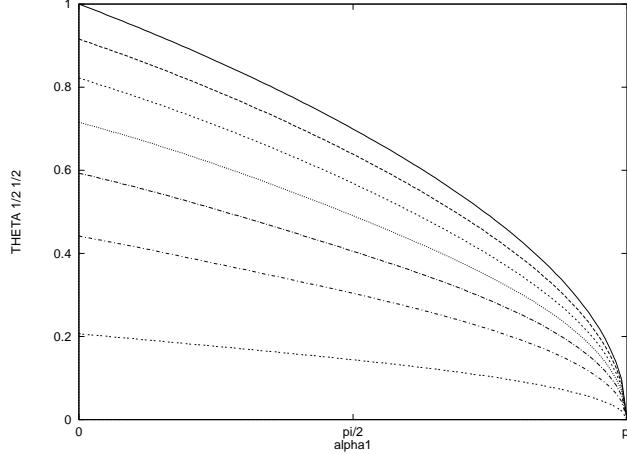


Figure 3: Correlation functions  $\Theta_{nn}(\alpha_1, \alpha_2)$  for  $\alpha_2 = 0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$  and  $n=1, n=2$  and  $n=1/2$  are shown on figures 3(a), 3(b), 3(c) respectively. The uppermost curve corresponds to  $\alpha_2 = 0$  for all three values of  $n$ .

Our numerical results indicate that the biggest coefficient in (26) is  $c_n$ .

Figure 3 shows the behaviour of the correlation function  $\Theta_{n,m}$  for various values of the  $n$  and  $m$ . Again the function  $\Theta_{n,m}$  does not simply coincide with

$$|\pi - \alpha_1|^n \cdot |\pi - \alpha_2|^m.$$

## 6 Comments

Finally we should stress that the curvature representation is very helpful in establishing global properties of the action and hence of the partition function and the loop Green functions [8, 16].

Indeed, with this representation it is easy to prove the lower estimate for the action  $A(M)$  [9] because

$$\tau = |\pi - \alpha^E|^{1/n} \geq \pi^{(1-n)/n} |\pi - \alpha^E|, \quad n \geq 1. \quad (27)$$

and for the extended curvature (17) we have

$$k_\tau(E) = \sum_{\langle ij \rangle} |\pi - \alpha_{ij}^E|^{1/n} \geq \pi^{(1-n)/n} \sum_{\langle ij \rangle} |\pi - \alpha_{ij}^E| \geq 2\pi^{1/n}, \quad (28)$$

where we have used the well known inequality for the total curvature of the polygon [15]. For the action this yields

$$A(M) = \int k_\tau(E) dE \geq 2\pi^{1/n} \int dE \geq 2\pi^{1/n} \Delta, \quad (29)$$

where  $\Delta$  is the diameter of the surface  $M$ .

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